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Degeneracy of Majorana bound states and fractional Josephson effect in a dirty SNS junction

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Abstract

We theoretically study the stability of more than one Majorana fermion appearing in a p-wave superconductor/dirty normal metal/p-wave superconductor junction in two-dimensions by using the chiral symmetry of a Hamiltonian. At the phase difference across the junction φ being π, we will show that all of the Majorana bound states in the normal metal belong to the same chirality. Due to this pure chiral feature, the Majorana bound states retain their high degree of degeneracy at zero energy even in the presence of a random potential. As a consequence, the resonant transmission of a Cooper pair via the degenerate Majorana bound states carries the Josephson current at $\varphi = \pi - 0^+$, which explains the fractional current-phase relationship discussed in a number of previous papers.

Keywords: unconventional superconductor, Majorana fermion, Josephson effect

(Some figures may appear in colour only in the online journal)

1. Introduction

The exotic properties of Majorana fermions (MFs) [1] have been a hot issue in condensed matter physics. MFs emerge as the surface bound states of topologically nontrivial superconductors such as a p-wave superconductor [2–4], a topological insulator/superconductor heterostructure [5], a spin–orbit coupled semiconductor/superconductor heterostructure [6–11] and a Shiba chain [12, 13]. As such the Majorana fermion bound states (MBSs) have attracted much attention from the view of the fault-tolerant topological quantum computation [14, 15]. Thus, the realization of MBSs is a recent desired subject in experimental fields [16–18].

When two one-dimensional semi-infinite p-wave superconductors are joined in a superconductor/insulator/superconductor (SIS) junction, a pair of MFs staying at the two junction interfaces form the Andreev bound states. As a consequence, the Josephson current exhibits the fractional current-phase (J-φ) relationship of $J(\varphi) \propto \sin(\varphi/2)$ at the zero temperature [3, 19]. The fractional Josephson effect is especially important because the effect provides a read-out process in the fault-tolerant topological computation [15]. Here we note that J is always 2π periodic in the direct-current Josephson effect. Thus the fractional current-phase relationship (CPR) means that the current jumps at $\varphi = \pm\pi$. It has been well known that a ballistic superconductor/normal metal/superconductor junction with the spin-singlet s-wave pairing symmetry [20–24] and a SIS junction with the spin-singlet d_{xy}-wave pairing symmetry [25–29] also indicate the fractional CPR. The unique feature to p-wave junctions is the persistence of the fractional CPR even in the presence of random impurity potential [30]. In fact, a theoretical study [31, 32] reported the fractional Josephson effect in a two-dimensional p_x-wave superconductor/dirty normal metal/p_x-wave superconductor (SNS) junction. In the two-dimensional junction, more than one MF degenerates at the zero-energy in the dirty normal metal and assist the resonant transmission of the Cooper pair at $\varphi = \pi - 0^+$. Generally speaking, the large degree of degeneracy in quantum states is

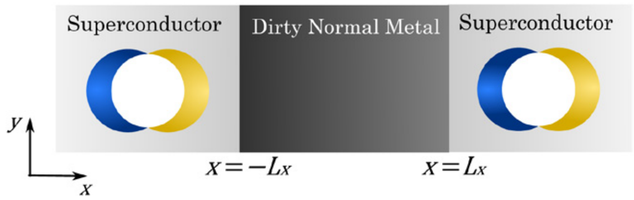

Figure 1. Schematic image of the p_x-wave superconductor/dirty normal metal/the p_x-wave superconductor junction.

a result of the high symmetry of the Hamiltonian. However it has been unclear what symmetry protects the degeneracy of the MBSs in a dirty normal metal. We address this issue in the present paper.

Several previous studies have suggested that chiral symmetry of Hamiltonian is a key feature to explain the stability of more than one MF at a surface of topologically nontrivial superconductors [33–37]. On the basis of these novel insights, we will prove the robustness of the degenerate MBSs in diffusive SNS junctions. In addition, we reconsider the meaning of a phenomenological theory of the fractional Josephson effect, where the tunneling Hamiltonian between the two edges at either sides of the insulator is described by $H_{\mathrm{T}} = -\mathrm{i}t\cos(\varphi/2)\gamma_{\mathrm{L}}\gamma_{\mathrm{R}}$ [3, 19]. Here t is the tunneling amplitude and γ_{L} (γ_{R}) is the operator of a MF at the edge of the superconductor on the left (right)-hand side of the insulator. The Josephson current calculated from $J \propto \partial_\varphi \langle H_{\mathrm{T}} \rangle$ exhibits the fractional CPR. However, this argument may be self-contradicted. The Josephson current flows at $\varphi = \pi - 0^+$ while the tunneling Hamiltonian vanishes. We also try to solve this puzzle in the present paper.

2. Chiral symmetry

Let us consider a two-dimensional SNS junction where two superconductors are characterized by an equal-spin-triplet p_x-wave symmetry as shown figure 1. The junction consists the three segments: a dirty normal metal ($-L_x \leqslant x \leqslant L_x$), and two superconductors ($L_x \leqslant j \leqslant \infty$ and $-\infty \leqslant j \leqslant -L_x$). The junction is described by the Bogoliubov-de Gennes Hamiltonian

$$H(\varphi) = H_{\mathrm{L}} + H_{\mathrm{N}} + H_{\mathrm{R}}(\varphi), \tag{1}$$

$$H_{\mathrm{L}} = \begin{bmatrix} \xi(\boldsymbol{r}) & \dfrac{\Delta}{k_{\mathrm{F}}}\partial_x \\ -\dfrac{\Delta}{k_{\mathrm{F}}}\partial_x & -\xi(\boldsymbol{r}) \end{bmatrix}, \tag{2}$$

$$H_{\mathrm{R}}(\varphi) = \begin{bmatrix} \xi(\boldsymbol{r}) & \dfrac{\Delta \mathrm{e}^{\mathrm{i}\varphi}}{k_{\mathrm{F}}}\partial_x \\ -\dfrac{\Delta \mathrm{e}^{-\mathrm{i}\varphi}}{k_{\mathrm{F}}}\partial_x & -\xi(\boldsymbol{r}) \end{bmatrix}, \tag{3}$$

$$H_{\mathrm{N}} = \begin{bmatrix} \xi(\boldsymbol{r}) + V_{\mathrm{imp}}(\boldsymbol{r}) & 0 \\ 0 & -\xi(\boldsymbol{r}) - V_{\mathrm{imp}}(\boldsymbol{r}) \end{bmatrix}, \tag{4}$$

$$\xi(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2 - \mu, \qquad k_{\mathrm{F}} = \sqrt{2m\mu}/\hbar, \tag{5}$$

where m denotes the effective mass of an electron, μ is the chemical potential, and Δ denotes the amplitude of the pair potential. In what follows, we consider 2×2 BdG Hamiltonian for one spin sector. The phase difference between the two superconductors is denoted by φ. The random impurity potential in the normal segment is represented by $V_{\mathrm{imp}}(\boldsymbol{r})$.

It is easy to confirm the following relations,

$$\Gamma H_{\mathrm{L}} \Gamma^{-1} = -H_{\mathrm{L}}, \tag{6}$$

$$(\mathrm{e}^{\mathrm{i}\varphi T_3}\Gamma)\, H_{\mathrm{R}}\, (\mathrm{e}^{\mathrm{i}\varphi T_3}\Gamma)^{-1} = -H_{\mathrm{R}}, \tag{7}$$

$$\Gamma H_{\mathrm{N}} \Gamma^{-1} = -H_{\mathrm{N}}, \tag{8}$$

$$(\mathrm{e}^{\mathrm{i}\varphi T_3}\Gamma)\, H_{\mathrm{N}}\, (\mathrm{e}^{\mathrm{i}\varphi T_3}\Gamma)^{-1} = -H_{\mathrm{N}}, \tag{9}$$

$$\Gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T_1, \qquad T_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{10}$$

Equation (6) represents chiral symmetry of H_{L} with respect to Γ. In the same way, equation (7) represents chiral symmetry of H_{R} with respect to $\mathrm{e}^{\mathrm{i}\varphi T_3}\Gamma$. The Hamiltonian in the normal part H_{N} preserves chiral symmetry for both Γ and $\mathrm{e}^{\mathrm{i}\varphi T_3}\Gamma$. When a Hamiltonian preserves chiral symmetry, the eigenstates of the Hamiltonian have two important features [38]. In the case of equation (6), for instance, one can prove following properties of eigen states of H_{L}.

(i) The eigenstates of the H_{L} at the zero energy are simultaneously the eigenstates of Γ with its eigenvalue (chirality) either $\gamma = +1$ or -1.

(ii) On the other hand, the nonzero-energy states of H_{L} are described by the linear combination of two different eigenstates of Γ: one has $\gamma = +1$ and the other has $\gamma = -1$.

Below we prove the stability of the highly degenerate zero energy states appearing in the SNS junction by taking these features into account. We note that the total Hamiltonian H preserves $\Gamma\, H \Gamma^{-1} = -H$ for φ being either 0 or $\pm\pi$.

We first analyze the chiral property of the zero-energy states appealing at the surface of the two semi-infinite superconductors ($x \leqslant -L_x$ and $x \geqslant L_x$). To do this, we remove the normal segment ($-L_x \leqslant x \leqslant L_x$) and apply the hard-wall boundary condition at $x = -L_x$ and $x = L_x$. In the y direction, the width of the superconductors is W and the hard-wall boundary condition is applied. By solving the Bogoliubov-de Gennes equation, we obtain the wave function for the the zero-energy states as

$$\psi_{L,n}(\boldsymbol{r}) = \frac{1}{\sqrt{N_n}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} X_{n,+}(x) Y_n(y), \tag{11}$$

$$\psi_{R,n}(\boldsymbol{r}) = \frac{1}{\sqrt{N_n}} \begin{bmatrix} \mathrm{e}^{\mathrm{i}\frac{\varphi}{2}} \\ -\mathrm{e}^{-\mathrm{i}\frac{\varphi}{2}} \end{bmatrix} X_{n,-}(x) Y_n(y), \tag{12}$$

$$X_{n,\pm}(x) = \sin[q_n(x \pm L_x)]\mathrm{e}^{\pm x/\xi}, \tag{13}$$

$$Y_n(y) = \sqrt{\frac{2}{W}}\sin\left(\frac{n\pi}{W}y\right), \tag{14}$$

$$q_n = \sqrt{k_n^2 - \xi^{-2}}, \quad \xi = \hbar^2 k_{\mathrm{F}}/m\Delta_0, \tag{15}$$

$$k_n = \frac{\sqrt{2m\mu_n}}{\hbar}, \qquad \mu_n = \mu - \frac{\hbar^2}{2m}\left(\frac{n\pi}{W}\right)^2, \tag{16}$$

where n indicates the propagating channels. The wave function $\psi_{L,n}$ ($\psi_{R,n}$) represents the nth zero-energy state localized at the surface of the left (right) superconductor. The normalization coefficient is denoted by N_n. The degree of degeneracy at the zero energy is equal to the number of the propagating channels N_c because a zero-energy state can be defined for each propagating channel. The derivations of the wave functions are shown in appendix. As indicated by the property (i), the zero-energy states in equations (11) and (13) are the eigenstates of Γ and $\mathrm{e}^{\mathrm{i}\varphi T_3}\Gamma$, respectively.

The particle-hole symmetry of the total Hamiltonian is represented by

$$\Xi\, H\, \Xi^{-1} = -H, \tag{17}$$

$$\Xi = \Gamma\, \mathcal{K}, \tag{18}$$

where $\mathcal{K}$ denotes the complex conjugation. Since

$$\Xi\psi_{L,n} = \psi_{L,n}, \tag{19}$$

$$\Xi\psi_{R,n} = -\psi_{R,n}, \tag{20}$$

all the zero-energy states are the Majorana bound states. Thus, at a surface of a p_x-wave superconductor, the degree of the degeneracy in MBSs is N_c.

3. Zero-energy states in SNS junctions

To analyze the MBSs in a SNS junction, we insert a normal segment described by H_{N} into the two superconductors. At $\varphi = 0$, the wave function $\psi_{L,n}$ and $\psi_{R,n}$ satisfies

$$\Gamma\psi_{L,n} = \psi_{L,n} \tag{21}$$

$$\Gamma\psi_{R,n} = -\psi_{R,n}, \tag{22}$$

for all n. Namely, all the MBSs in the left superconductor belong to $\gamma = +1$ while those in the right superconductor belong to $\gamma = -1$ as shown in figure 2(a). The MBSs at the surface of the two different superconductors have the opposite chirality to each other. In a SNS junction, a normal metal connects the two superconductor. MBSs with $\gamma = +1$ (MBSs with $\gamma = -1$) penetrate into the normal metal from the left (right) superconductor. As a result, they form nonzero-energy states there. In this way, the penetration of MBSs into the normal metal lifts the high degeneracy at the zero-energy. In other words, pairs of MFs couple-back to conventional quasiparticles and the number of such pairs is N_c.

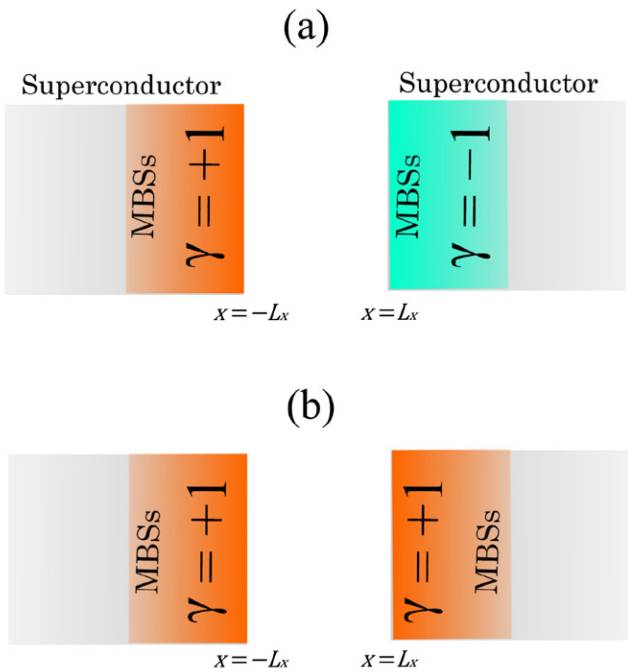

Figure 2. Schematic image for the chirality of the Majorana bound states. (a) At $\varphi = 0$, the left-side MBSs and the right-side MBSs have the opposite chirality to each other. (b) On the other hand, at $\varphi = \pi$, both left-side and right-side MBSs have the chirality $\gamma = +1$.

On the other hand at $\varphi = \pi$, one can find

$$\Gamma\psi_{L,n} = \psi_{L,n} \tag{23}$$

$$\Gamma\psi_{R,n} = \psi_{R,n}. \tag{24}$$

Both $\psi_{L,n}$ and $\psi_{R,n}$ belong to the same chirality $\gamma = +1$ as shown in figure 2(b). The MBSs retain their high degree of degeneracy even in a SNS junction because the zero-energy states with $\gamma = -1$ are absent in the normal metal. According to the property (ii), the zero-energy states belonging the same chirality cannot form any nonzero-energy states.

To confirm the argument above, we calculate the wave function in a SNS junction. We first set the impurity potential $V_{\mathrm{imp}}(\boldsymbol{r}) = 0$ and solve the Bogoliubov-de Gennes equation at the zero energy for $\varphi = \pi$,

$$H(\pi)\psi_0 = 0. \tag{25}$$

A solution of equation (25) is given by (see also appendix)

$$\psi'_{L,n}(\boldsymbol{r}) = \begin{bmatrix}1\\1\end{bmatrix}[a_L\mathrm{e}^{\mathrm{i}q_n x} + b_L\mathrm{e}^{-\mathrm{i}q_n x}]\mathrm{e}^{x/\xi}Y_n(y), \tag{26}$$

$$\psi_{N,n}(\boldsymbol{r}) = \left[\begin{bmatrix}a_N\\c_N\end{bmatrix}\mathrm{e}^{\mathrm{i}k_n x} + \begin{bmatrix}b_N\\d_N\end{bmatrix}\mathrm{e}^{-\mathrm{i}k_n x}\right]Y_n(y), \tag{27}$$

$$\psi'_{R,n}(\boldsymbol{r}) = \begin{bmatrix}1\\1\end{bmatrix}[a_R\mathrm{e}^{\mathrm{i}q_n x} + b_R e^{-\mathrm{i}q_n x}]\,\mathrm{e}^{-x/\xi}Y_n(y), \tag{28}$$

where $\psi'_{L,n}$, $\psi_{N,n}$, and $\psi'_{R,n}$ are the wave function at the n-th propagating channel in the left superconductor, the normal metal, and the right superconductor, respectively. By reflecting the chiral property of the MBSs in two superconductors, the vector structure of the wave functions in the superconducting segments takes the particular form of $\psi'_{L(R),n} \propto [1, 1]^{\mathrm{T}}$. By applying the boundary condition at the two interfaces, we obtain the two orthogonal zero-energy states for each propagating channel as

$$\psi_{\pm} = \frac{1}{\sqrt{N_{\pm}}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \phi_{n,\pm}(x) Y_n(y), \tag{29}$$

$$\phi_{n,+}(x) = \begin{cases} A_+(x) & \text{for } x \leqslant -L_x \\ \sin(k_n x) & \text{for } -L_x \leqslant x \leqslant L_x \\ B_+(x) & \text{for } x \geqslant L_x, \end{cases} \tag{30}$$

$$\phi_{n,-}(x) = \begin{cases} A_-(x) & \text{for } x \leqslant -L_x \\ \cos(k_n x) & \text{for } -L_x \leqslant x \leqslant L_x \\ B_-(x) & \text{for } x \geqslant L_x, \end{cases} \tag{31}$$

$$A_{\pm}(x) = c_{\pm} \sin\{q_n(x + L_x) \mp \theta_{\pm}\} \mathrm{e}^{(x+L_x)/\xi}, \tag{32}$$

$$B_{\pm}(x) = c_{\pm} \sin\{q_n(x - L_x) \pm \theta_{\pm}\} \mathrm{e}^{-(x-L_x)/\xi}, \tag{33}$$

$$c_{\pm} = \frac{\sqrt{k_n\{k_n \pm \xi^{-1} \sin(2k_n L_x)\}}}{q_n}, \tag{34}$$

$$\theta_{+} = \arctan\left[\frac{q_n \sin(k_n L_x)}{k_n \cos(k_n L_x) + \xi^{-1} \sin(k_n L_x)}\right], \tag{35}$$

$$\theta_{-} = \arctan\left[\frac{q_n \cos(k_n L_x)}{k_n \sin(k_n L_x) - \xi^{-1} \cos(k_n L_x)}\right], \tag{36}$$

where $N_{\pm}$ is a normalization coefficient. Since we obtain the two zero-energy states for each propagating channel, the degeneracy of the zero-energy bound states becomes twice the number of the propagating channel N_c. More importantly, equation (29) suggests that all the zero-energy states in SNS junction are the eigenstates of Γ belonging to $\gamma = +1$.

Next we introduce the impurity potential V_{imp} into the normal segment. The random potential modifies the wave function in equation (29). Actually we cannot analytically describe how the wave function depends on $\boldsymbol{r}$ anymore. But the vector part of the wave function $[1, 1]^{\mathrm{T}}$ remains unchanged even in the presence of impurity potentials because V_{imp} preserves the chiral symmetry. Therefore all the zero-energy states keep their chirality at $\gamma = +1$ even in the presence of V_{imp}. According to the property (ii), such chirality aligned zero-energy states keep their high degeneracy because they cannot construct nonzero-energy states in the absence of their chiral partner belonging to $\gamma = -1$. As a result, the degenerate MBSs form the resonant transmission channels in the normal metal. The Josephson current at $\varphi = \pi - 0^+$ flows through such highly degenerate resonant states. Our analysis provides a mathematical background for understanding the fractional Josephson effect in a dirty SNS junction which was numerically shown in the previous papers [31, 32].

4. Phenomenological theory

The fractional Josephson effect in one-dimensional SIS can be phenomenologically explained in terms of the effective hopping Hamiltonian between the two Majorana bound states. At the edge of isolated semi-infinite p-wave superconductor, the electron operators at the edges are described by

$$\Psi_L = \gamma_L, \tag{37}$$

$$\Psi_R = \mathrm{i}\mathrm{e}^{\mathrm{i}\varphi/2} \gamma_{\mathrm{R}}, \tag{38}$$

where γ_{L} (γ_{R}) is the operator of a Majorana fermion at the edge of left (right) superconductor. The tunneling Hamiltonian between the two edges becomes

$$H_{\mathrm{T}} = -t[\Psi_L^{\dagger}\Psi_R + \Psi_R^{\dagger}\Psi_L] \tag{39}$$

$$= -2\mathrm{i}t \cos(\varphi/2)\, \gamma_{\mathrm{L}}\, \gamma_{\mathrm{R}}. \tag{40}$$

The expectation value of the tunneling Hamiltonian could be

$$\langle H_{\mathrm{T}} \rangle = -2tC_0 \cos(\varphi/2), \tag{41}$$

where we assume that $\langle \mathrm{i}\gamma_{\mathrm{L}}\, \gamma_{\mathrm{R}} \rangle = C_0$ is a constant. The Josephson current calculated as

$$J = \frac{e}{\hbar} \partial_{\varphi} \langle H_{\mathrm{T}} \rangle = \frac{e2tC_0}{\hbar} \sin(\varphi/2), \tag{42}$$

describes the fractional current-phase relationship. At $\varphi = \pi - 0^+$, we obtain

$$\langle H_{\mathrm{T}} \rangle = -tC_0\, 0^+, \tag{43}$$

$$J = \frac{e2tC_0}{\hbar}\left(1 - \frac{(0^+)^2}{8}\right). \tag{44}$$

The Josephson current takes its maximum, whereas the amplitude of the tunneling Hamiltonian is proportional to 0^+. The Josephson current at $\varphi = \pi - 0^+$ flows as a result of the resonant transmission through the junction. Therefore the amplitude of the current is not proportional to the amplitude of the tunneling Hamiltonian. This argument is valid as far as $\varphi = \pi - 0^+$. At $\varphi = \pi$, the tunneling Hamiltonian vanishes exactly, which leads to the absence of the Josephson current. In this way, the phenomenological argument using equation (39) is consistent with the microscopic theory of the fractional Josephson effect.

5. Conclusion

We have studied the stability of more than one Majorana Fermion appearing in a two-dimensional superconductor/normal metal/superconductor (SNS) junction in terms of chiral symmetry of Hamiltonian, where the two superconductors are characterized by spin-triplet p_x-wave symmetry. When the phase difference across the junction φ is either 0 or π, the Hamiltonian of the SNS junction preserves chiral symmetry. At $\varphi = \pi$, the Majorana bound states (MBSs) in the normal metal can retain their high degree of degeneracy at the zero energy even in the presence of the impurity scatterings because all of the MBSs belong to the same chirality. As a consequence, the resonant transmission of a Cooper pair via such highly degenerate MBSs carries the Josephson current at $\varphi = \pi - 0^+$. The physical picture obtained in this paper explains well the persistence of the fractional current-phase relationship in a dirty SNS junction which was numerically shown in previous papers. We have also discussed a way of understanding the fractional current-phase relationship derived from a phenomenological tunneling Hamiltonian of a Majorana Fermion.

Acknowledgments

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Appendix. Wave function of zero energy states

We derive the wave functions of the zero-energy states appearing in the two semi-infinite p_x-wave superconductors illustrated in figure 1. In the y direction, the width is denoted by W and the hard-wall boundary condition is applied. The Bogoliubov-de Gennes equation is given by

$$H_\alpha \varphi_{\alpha,E} = E\varphi_{\alpha,E}, \tag{A.1}$$

$$H_\alpha = \begin{bmatrix} \xi(\boldsymbol{r}) & \dfrac{\Delta e^{i\varphi_\alpha}}{k_F}\partial_x \\ -\dfrac{\Delta e^{-i\varphi_\alpha}}{k_F}\partial_x & -\xi(\boldsymbol{r}) \end{bmatrix}, \tag{A.2}$$

where the index $\alpha = L,R$ labels the left superconductor ($x \leqslant -L_x$) and the right superconductor ($x \geqslant L_x$), respectively. The superconducting phase is given as

$$\varphi_L = 0, \tag{A.3}$$

$$\varphi_R = \varphi. \tag{A.4}$$

The Hamiltonian H_α preserves chiral symmetry as

$$(e^{i\varphi_\alpha T_3}\Gamma)\, H_\alpha\, (e^{i\varphi_\alpha T_3}\Gamma)^{-1} = -H_\alpha, \tag{A.5}$$

$$\Gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T_1, \quad T_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.6}$$

By solving equation (A.1), we obtain the wave function belonging to an energy E as

$$\psi_{\alpha,E} = \phi_n(x) Y_n(y) \tag{A.7}$$

$$\begin{aligned} \phi_n(x) = {} & c_1 \begin{bmatrix} E+\Omega_+ \\ -i\Delta e^{-i\varphi_\alpha}(k_+/k_F) \end{bmatrix} e^{ik_+x} \\ & + c_2 \begin{bmatrix} i\Delta e^{i\varphi_\alpha}(k_-/k_F) \\ E+\Omega_- \end{bmatrix} e^{ik_-x} \\ & + c_3 \begin{bmatrix} -i\Delta e^{i\varphi_\alpha}(k_-/k_F) \\ E+\Omega_- \end{bmatrix} e^{-ik_-x} \\ & + c_4 \begin{bmatrix} E+\Omega_+ \\ i\Delta e^{-i\varphi_\alpha}(k_+/k_F) \end{bmatrix} e^{-ik_+x}, \end{aligned} \tag{A.8}$$

$$Y_n(y) = \sqrt{\frac{2}{W}}\sin\left(\frac{n\pi}{W}y\right), \tag{A.9}$$

$$\Omega_\pm = \sqrt{E^2 - \frac{\Delta^2}{\mu}\left(\mu_n - \frac{\Delta^2}{4\mu}\right)} \mp \frac{\Delta^2}{2\mu}, \tag{A.10}$$

$$k_\pm = k_n\sqrt{1 \pm \frac{\Omega_\pm}{\mu_n}}, \qquad k_n = \frac{\sqrt{2m\mu_n}}{\hbar}, \tag{A.11}$$

$$\mu_n = \mu - \frac{\hbar^2}{2m}\left(\frac{n\pi}{W}\right)^2, \tag{A.12}$$

where c_i ($i = 1 - 4$) are numerical coefficients. At $E = 0$, the wave function is deformed as

$$\psi_{\alpha,0}(x,y) = [\phi_{\alpha,n,1}(x) + \phi_{\alpha,n,2}(x)]\, Y_n(y), \tag{A.13}$$

$$\phi_{\alpha,n,1}(x) = \begin{bmatrix} e^{i\frac{\varphi_\alpha}{2}} \\ -e^{-i\frac{\varphi_\alpha}{2}} \end{bmatrix} [c_1' e^{iq_n x} + c_3' e^{-iq_n x}]\, e^{-x/\xi}, \tag{A.14}$$

$$\phi_{\alpha,n,2}(x) = \begin{bmatrix} e^{i\frac{\varphi_\alpha}{2}} \\ e^{-i\frac{\varphi_\alpha}{2}} \end{bmatrix} [c_2' e^{iq_n x} + c_4' e^{-iq_n x}]\, e^{x/\xi}, \tag{A.15}$$

$$q_n = \sqrt{k_n^2 - \xi^{-2}}, \qquad \xi = \hbar^2 k_F / m\Delta_0. \tag{A.16}$$

We note that the components of $\phi_{\alpha,n,1}$ and $\phi_{\alpha,n,2}$ are the eigenstates of the chiral symmetry operator as

$$(e^{i\varphi_\alpha T_3}\Gamma)\phi_{\alpha,n,1}(x) = -\phi_{\alpha,n,1}(x), \tag{A.17}$$

$$(e^{i\varphi_\alpha T_3}\Gamma)\phi_{\alpha,n,2}(x) = \phi_{\alpha,n,2}(x). \tag{A.18}$$

First, we calculate the wave function of the zero-energy states in the left superconductor. We apply the boundary condition in the x direction as

$$\psi_{L,0}(-\infty, y) = \psi_{L,0}(-L_x, y) = 0. \tag{A.19}$$

As a result, we obtain the two zero-energy states for each propagating channel as

$$\psi_L = \frac{1}{\sqrt{N_n}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sin[q_n(x + L_x)] e^{x/\xi} Y_n(y), \tag{A.20}$$

where the normalization coefficient is denoted by N_n. In is easy to show that the zero energy states of left superconductor are simultaneously the eigenstates of chiral symmetry operator Γ with the eigenvalue $\gamma = +1$. Next, we consider the right superconductor. By applying the boundary condition in the x direction as

$$\psi_{R,0}(L_x, y) = \psi_{R,0}(\infty, y) = 0, \tag{A.21}$$

we find the wave function for the zero-energy states as

$$\psi_R = \frac{1}{\sqrt{N_n}} \begin{bmatrix} e^{i\frac{\varphi}{2}} \\ -e^{-i\frac{\varphi}{2}} \end{bmatrix} \sin[q_n(x - L_x)] e^{-x/\xi} Y_n(y). \tag{A.22}$$

The zero-energy states of the right superconductor ψ_R hold $\gamma = -1$ for the chiral symmetry operator $e^{i\varphi T_3}\Gamma$.

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